

# Order and minimality of some topological groups <sup>☆</sup>

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## Abstract

A Hausdorff topological group is called minimal if it does not admit a strictly coarser Hausdorff group topology. This paper mostly deals with the topological group  $H_+(X)$  of order-preserving homeomorphisms of a compact linearly ordered connected space  $X$ . We provide a sufficient condition on  $X$  under which the topological group  $H_+(X)$  is minimal. This condition is satisfied, for example, by: the unit interval, the ordered square, the extended long line and the circle (endowed with its cyclic order). In fact, these groups are even  $a$ -minimal, meaning, in this setting, that the compact-open topology on  $G$  is the smallest Hausdorff group topology on  $G$ . One of the key ideas is to verify that for such  $X$  the Zariski and the Markov topologies on the group  $H_+(X)$  coincide with the compact-open topology. The technique in this article is mainly based on a work of Gartside and Glyn [22].

*Keywords:*  $a$ -minimal group, Markov's topology, minimal groups, compactLOTS, order-preserving homeomorphisms, Zariski's topology.

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## 1. Introduction

A Hausdorff topological group  $G$  is *minimal* ([17], [38]) if it does not admit a strictly coarser Hausdorff group topology or, equivalently, if every injective continuous group homomorphism  $G \rightarrow P$  into a Hausdorff topological group is a topological group embedding.

All topological spaces are assumed to be Hausdorff and completely regular (unless stated otherwise). Let  $X$  be a compact topological space. Denote by  $H(X)$  the group of all homeomorphisms of  $X$ , endowed with the compact-open

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topology  $\tau_{co}$ . In this setting  $H(X)$  is a topological group and the natural action  $H(X) \times X \rightarrow X$  is continuous.

Clearly, every compact topological group is minimal. The groups  $\mathbb{R}$  and  $\mathbb{Z}$ , on the other hand, are not minimal. Moreover, Stephenson showed in [38] that an LCA group is minimal if and only if it is compact. Nontrivial examples of minimal groups include  $\mathbb{Q}/\mathbb{Z}$  with the quotient topology, [38], and  $S(X)$ , the symmetric group of an infinite set (with the pointwise topology). The minimality of the latter was proved by Gaughan [23] and (independently) by Dierolf and Schwanengel [9]. For more information on minimal groups we refer to the surveys [8], [10], [11] and the book [12].

The following is a question of Stoyanov (cited in [1], for example):

*Question 1.1.* (Stoyanov) Is it true that for every compact homogeneous space  $X$  the topological group  $H(X)$  is minimal?

One important positive example of such a space is the Cantor cube  $2^\omega$ . Indeed, in [21] Gamarnik proved that  $H(2^\omega)$  is minimal. Recently van Mill ([30]) provided a counterexample to Question 1.1 proving that for the  $n$ -dimensional Menger universal continuum  $X$ , where  $n > 0$ , the group  $H(X)$  is not minimal.

It is well known that the Hilbert cube  $[0, 1]^\omega$  is a homogeneous compact space as well. The following question of Uspenskiĭ [41] remains unanswered: is the group  $H([0, 1]^\omega)$  minimal?

**Definition 1.2.**

1. [11] A topological group  $G$  is *a-minimal* if its topology is the smallest possible Hausdorff group topology on  $G$ .
2. [11] A compact space  $X$  is *M-compact* (*aM-compact*) if the topological group  $H(X)$  is minimal (respectively, *a-minimal*).
3. A compact ordered space  $X$  is *M<sub>+</sub>-compact* (*aM<sub>+</sub>-compact*) if the topological group  $H_+(X)$  of all order-preserving homeomorphisms of  $X$  is minimal (respectively, *a-minimal*).

Several questions naturally arise at this point:

*Question 1.3.*

1. [11] Which (notable) compact spaces are *M-compact*? *aM-compact*?
2. Which compact ordered spaces are *M<sub>+</sub>-compact*? *aM<sub>+</sub>-compact*?

The two point compactification of  $\mathbb{Z}$  is a compact LOTS  $X$  such that  $H_+(X)$  and  $H(X)$  are not minimal (Example 4.1). Thus not every compact LOTS is *M<sub>+</sub>-compact* or *M-compact*.

Clearly, every *a-minimal* group is minimal. It is well known that  $(\mathbb{Z}, \tau_p)$  with its  $p$ -adic topology is a minimal topological group. Since such topologies are incomparable for different  $p$ 's, it follows that  $(\mathbb{Z}, \tau_p)$  is not *a-minimal*.

Recall a few results:

1. (Gaughan [23]) The symmetric group  $S(X)$  is  $a$ -minimal. Since  $H(X^*)$  is precisely  $S(X)$  we obtain that the 1-point compactification  $X^*$  of a discrete set  $X$  is  $aM$ -compact.
2. (Banakh-Guran-Protasov [2]) Every subgroup of  $S(X)$  that contains  $S_\omega(X)$  (permutations of finite support) is  $a$ -minimal (answers a question of Dikranjan [29]).
3. (Gamarnik [21])  $[0, 1]^n$  is  $M$ -compact (for  $n \in \mathbb{N}$ ) if and only if  $n = 1$ .
4. (Gartside and Glyn [22])  $[0, 1]$  and  $\mathbb{S}^1$  are  $aM$ -compact.
5. (Gamarnik [21]) The Cantor cube  $2^\omega$  is  $M$ -compact.
6. (Uspenskij [40]) Every  $h$ -homogeneous compact space is  $M$ -compact.
7. (van Mill [30])  $n$ -dimensional Menger universal continuum  $X$ , where  $n > 0$ , is not  $M$ -compact (answers Stoyanov's Question 1.1).

Recall that a zero-dimensional compact space  $X$  is  $h$ -homogeneous if all non-empty clopen subsets of  $X$  are homeomorphic to  $X$ . In particular,  $2^\omega$  is  $h$ -homogeneous. Hence, (6) is a generalization of (5).

The concept of an  $a$ -minimal group is in fact an intrinsic algebraic property of an abstract group  $G$  (underlying a given topological group).  $a$ -minimality is interesting for several reasons. For instance, it is strongly related to some fundamental topics like Markov's and Zariski's topologies.

For additional information about  $a$ -minimality (and minimality) see the recent survey [11]. For Markov's and Zariski's topologies see [15], [14], [2], [16]. We recall the definitions.

**Definition 1.4.** Let  $G$  be a group.

1. The *Zariski topology*  $\mathfrak{Z}_G$  is generated by the sub-base consisting of the sets  $\{x \in G : x^{\varepsilon_1} g_1 x^{\varepsilon_2} g_2 \cdots x^{\varepsilon_n} g_n \neq e\}$ , where  $e$  is the unit element of  $G$ ,  $n \in \mathbb{N}$ ,  $g_1, \dots, g_n \in G$ , and  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ .
2. The *Markov topology*  $\mathfrak{M}_G$  is the infimum (taken in the lattice of all topologies on  $G$ ) of all *Hausdorff* group topologies on  $G$ .

Note that  $(G, \mathfrak{Z}_G)$  and  $(G, \mathfrak{M}_G)$  are quasi-topological groups. That is the inverse and the translations are continuous. They are not necessarily topological groups. In fact, if  $G$  is abelian then  $\mathfrak{Z}_G$  and  $\mathfrak{M}_G$  are not group topologies, unless  $G$  is finite, [14, Corollary 3.6]. Here we give some simple properties. Regarding assertion (3) in the following lemma see for example [11, Definition 2.1].

**Lemma 1.5.** *Let  $G$  be an abstract group. Suppose that  $\tau$  is a Hausdorff group topology on  $G$ . Then*

- (1)  $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \tau$ .
- (2)  $\mathfrak{Z}_G = \mathfrak{M}_G = \tau$  if and only if  $\tau \subseteq \mathfrak{Z}_G$ . In this case  $(G, \tau)$  is  $a$ -minimal.

- (3)  $\mathfrak{M}_G$  is a (not necessarily, Hausdorff) group topology if and only if  $(G, \mathfrak{M}_G)$  is an  $\alpha$ -minimal topological group.

*Proof.* (1) Follows directly from the definitions.

(2) Follows from (1).

- (3) Note that  $\mathfrak{M}_G$  is always a  $T_1$ -topology. Hence, if  $\mathfrak{M}_G$  is a topological group topology then it is Hausdorff. Taking into account the definition of  $\mathfrak{M}_G$  we can conclude that this topology is the smallest Hausdorff group topology on  $G$ . Hence,  $(G, \mathfrak{M}_G)$  is  $\alpha$ -minimal.

□

*Question 1.6.* [Markov] For what groups  $G$  the Markov and Zariski topologies coincide?

A review of some old and new partial answers can be found in [16]. Below, in Theorem 3.4, we give additional examples of groups for which  $\mathfrak{Z}_G = \mathfrak{M}_G$ .

In the present paper we mainly deal with the groups  $H_+(X)$ . Given an ordered compact space  $X$ , we are interested in the group  $H_+(X)$  of order-preserving homeomorphisms. For a compact space  $X$  the group  $H(X)$  is complete (with respect to the two-sided uniformity) and therefore  $H_+(X)$  is also complete (as a closed subgroup of a complete group).

In certain cases the minimality of  $H(X)$  can be deduced from the minimality of  $H_+(X)$ , as the following lemma shows.

**Lemma 1.7.** *Let  $X$  be a compact LOTS such that  $H_+(X)$  is minimal. If  $H_+(X)$  is a co-compact subgroup of  $H(X)$ , then  $H(X)$  is minimal.*

This lemma is a corollary of Lemma 2.2. *Co-compactness* of  $H_+(X)$  in  $H(X)$  means that the coset space  $H(X)/H_+(X)$  is compact.

If  $X$  is a linearly ordered continuum, then by Lemma 2.3 the subgroup  $H_+(X)$  has at most index 2 in  $H(X)$ . So, in this case, from the minimality of  $H_+(X)$  we can deduce by Lemma 1.7 the minimality of  $H(X)$ . For example, it is true for  $X = [0, 1]$ . Note that  $H[0, 1] = H_+[0, 1] \rtimes \mathbb{Z}_2$ , the topological semidirect product of  $H_+[0, 1]$  and  $\mathbb{Z}_2$ , where  $\mathbb{Z}_2$  is the two element group. However, in general, it is unclear how to infer the minimality of a topological group  $G$  from the minimality of  $G \rtimes \mathbb{Z}_2$ . For instance, in [11, Example 4.7] it is shown that there exists a non-minimal group  $G$  such that  $G \rtimes \mathbb{Z}_2$  is minimal.

Recall the following result of Gartside and Glyn:

**Theorem 1.8.** [22] *For any metric one dimensional manifold (with or without boundary)  $M$ , the compact-open topology on the full homeomorphism group  $H(M)$  is the unique minimum Hausdorff group topology on  $H(M)$ .*

The one dimensional compact manifolds, up to homeomorphism, are the closed interval  $[0, 1]$  and the circle  $\mathbb{S}^1$ . In view of Definition 1.2 this result implies the following.

**Theorem 1.9.** [22]  $H[0, 1]$  and  $H(S^1)$  are  $a$ -minimal groups.

Extending some ideas of Gartside-Glyn [22] to linearly ordered spaces we give some new results about minimality of the groups  $H_+(X)$  of order preserving homeomorphisms.

**Theorem** (see Theorem 3.4). *Let  $(X, \tau_\leq)$  be a compact connected LOTS that satisfies the following condition:*

(A) *for every pair of elements  $a < b$  in  $X$  the group  $H_+[a, b]$  is nontrivial.*

*Then:*

- (1) *For the topological group  $G = H_+(X)$  and  $G = H(X)$  the Zariski and Markov topologies coincide with the compact-open topology. That is,  $\mathfrak{Z}_G = \mathfrak{M}_G = \tau_{co}$ .*
- (2) *The topological groups  $H_+(X)$  and  $H(X)$  are  $a$ -minimal.*
- (3)  *$X$  is  $aM_+$ -compact and  $aM$ -compact.*

According to results of Hart and van Mill [26] (see Section 4.2) there exists a connected compact LOTS  $X$  which is  $H_+$ -rigid, that is,  $H_+(X)$  is trivial (in fact,  $H(X)$  is trivial). Hence, condition (A) of the theorem above is not always satisfied for general ordered continua. Moreover, one may derive from results of [26] that there exists a connected compact LOTS  $X$  for which  $H_+(X) = H(X) = \mathbb{Z}$ , a discrete copy of the integers  $\mathbb{Z}$ , and  $H[c, d]$  is trivial for some pair  $c < d$  in  $X$  (Proposition 4.2).

In Section 4 we give some concrete examples of spaces that satisfy condition (A) of Theorem 3.4. The following linearly ordered spaces  $X$  are  $aM_+$ -compact, that is the groups  $H_+(X)$  are  $a$ -minimal:

- 1.  $[0, 1]$ ;
- 2. the lexicographically ordered square  $\mathcal{I}^2$ ;
- 3. the extended long line  $\mathcal{L}^*$ ;
- 4. the ordinal space  $[0, \kappa]$ ;
- 5. the unit circle  $S^1$  (in this case we work with a *cyclic order*, Definition 2.6).

Note that the groups  $H_+(X)$  play a major role in many research lines. See, for example, [24, 33, 25].

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## 2. Preliminaries

In what follows, every compact topological space will be considered as a uniform space with respect to its natural (unique) uniformity.

For a topological group  $(G, \gamma)$  and its subgroup  $H$  denote by  $\gamma/H$  the natural quotient topology on the coset space  $G/H$ .

**Lemma 2.1.** (Merson's Lemma) *Let  $(G, \gamma)$  be a not necessarily Hausdorff topological group and  $H$  be a not necessarily closed subgroup of  $G$ . If  $\gamma_1 \subseteq \gamma$  is a coarser group topology on  $G$  such that  $\gamma_1|_H = \gamma|_H$  and  $\gamma_1/H = \gamma/H$ , then  $\gamma_1 = \gamma$ .*

**Lemma 2.2.** *Let  $H$  be a co-compact complete subgroup of a topological group  $G$ . If  $H$  is minimal then  $G$  is minimal too.*

*Proof.* Denote by  $\tau$  the given topology on  $G$ , and let  $\gamma \subseteq \tau$  be a coarser Hausdorff group topology. Since  $H$  is minimal, we know that  $\gamma|_H = \tau|_H$ . Furthermore,  $H$  is  $\gamma$ -closed in  $G$  because  $H$  is complete. Since  $(G/H, \gamma/H)$  is Hausdorff and  $(G/H, \tau/H)$  is compact we have  $\gamma/H = \tau/H$ . Thus, by Merson's Lemma 2.1, we conclude that  $\gamma = \tau$ .  $\square$

### 2.1. Ordered topological spaces

A *linear order* on a set  $X$  is, as usual, a binary relation  $\leq$  which is reflexive, antisymmetric, transitive and satisfies in addition the totality axiom: for all  $a, b \in X$  either  $a \leq b$  or  $b \leq a$ .

For a set  $X$  equipped with a linear order  $\leq$ , the *order topology* (or *interval topology*)  $\tau_{\leq}$  on  $X$  is generated by the subbase that consists of the intervals  $(\leftarrow, a) = \{x \in X : x < a\}$ ,  $(b, \rightarrow) = \{x \in X : b < x\}$ . A *linearly ordered topological space* (or LOTS) is a triple  $(X, \tau_{\leq}, \leq)$  where  $\leq$  is a linear order on  $X$  and  $\tau_{\leq}$  is the order topology on  $X$ . For every pair  $a < b$  in  $X$  the definition of the intervals  $(a, b), [a, b]$  is understood. Every linearly ordered compact space  $X$  has the smallest and the greatest element; so,  $X = [s, t]$  for some  $s, t \in X$ .

Sometimes we say: *linearly ordered continuum*, instead of *compact and connected LOTS*.

**Lemma 2.3.** *Let  $(X, \tau_{\leq})$  be a linearly ordered continuum. Then every  $f \in H(X)$  is either order-preserving or order-reversing. In particular, the index of  $H_+(X)$  in  $H(X)$  is at most 2.*

*Proof.* Assume for contradiction that there exists  $f \in H(X)$  such that  $f$  is neither order-preserving nor order-reversing. Thus there exist three points  $x_1, x_2, x_3 \in X$  such that  $x_1 < x_2 < x_3$  and either  $f(x_1) < f(x_2) \wedge f(x_2) > f(x_3)$  or  $f(x_1) > f(x_2) \wedge f(x_2) < f(x_3)$ . Both cases lead to a contradiction. We give the details for the first case (the second case is similar).

Suppose  $x_1 < x_2 < x_3$  and  $f(x_1) < f(x_2) \wedge f(x_2) > f(x_3)$ . Since  $X$  is linearly ordered there are two possibilities to consider.

1.  $f(x_1) < f(x_3) < f(x_2)$ : then by the Intermediate Value Theorem (applied to the interval  $[x_1, x_2]$ ) there exists  $x_1 < x_0 < x_2$  such that  $f(x_0) = f(x_3)$ , which is a contradiction since  $f$  is 1-1.

2.  $f(x_3) < f(x_1) < f(x_2)$ : then again by the Intermediate Value Theorem (applied to the interval  $[x_2, x_3]$ ) there exists  $x_2 < x_0 < x_3$  such that  $f(x_0) = f(x_1)$ , which is a contradiction because  $f$  is 1-1.

Each case leads to a contradiction, and this fact concludes the proof.  $\square$

In the sequel we use several times the following simple "localization lemma".

**Lemma 2.4.** *Let  $X$  be a LOTS and let  $a < b$  be a given pair of elements in  $X$ . If  $h \in H_+[a, b]$ , then for the natural extension  $\hat{h}: X \rightarrow X$ , with  $\hat{h}(x) = x$  for every  $x \in X \setminus (a, b) = (\leftarrow, a] \cup [b, \rightarrow)$ , we have  $\hat{h} \in H_+(X)$ .*

The idea of the following lemma was kindly provided to us by K.P. Hart.

**Lemma 2.5.** *Let  $X$  be a linearly ordered continuum. The following conditions are equivalent:*

- (A) *for every pair of elements  $a < b$  in  $X$  the group  $H_+[a, b]$  is nontrivial,*
- (B) *for every pair of elements  $a < b$  in  $X$  the group  $H_+[a, b]$  is nonabelian.*

*Proof.* Let  $a < b$  in  $X$ . Assuming (A) there exists a nontrivial  $h_1 \in H_+[a, b]$ . So,  $h_1(u) \neq u$  for some  $u \in (a, b)$ . We can suppose that  $a < u < h_1(u) < b$  (indeed, if  $h_1(u) < u$ , replace  $h_1$  by  $h_1^{-1}$  and  $u$  by  $h_1(u)$ ). Since  $X$  is a continuum, the interval  $(u, h_1(u))$  is nonempty. Choose an arbitrary  $v \in (u, h_1(u))$ . By the continuity of  $h_1$  there exists a sufficiently small neighbourhood  $O$  of  $u$  such that

$$s < v < h_1(t)$$

for every  $s, t \in O$ . Without restriction of generality we can assume that  $O$  is the interval  $[x_1, x_2]$ , where  $x_1 < x_2$ . Clearly,  $h_1(x_1) < h_1(x_2)$ , so

$$a < x_1 < x_2 < h_1(x_1) < h_1(x_2) < b.$$

Now apply condition (A) to the interval  $[h_1(x_1), h_1(x_2)]$ . There exists a nontrivial  $h_2 \in H_+[h_1(x_1), h_1(x_2)]$ . Similarly, as for  $h_1$  and  $[a, b]$ , one may choose, for  $h_2$  and  $[h_1(x_1), h_1(x_2)]$ , a subinterval  $[y_1, y_2]$  of  $[h_1(x_1), h_1(x_2)]$  such that

$$h_1(x_1) < y_1 < y_2 < h_2(y_1) < h_2(y_2) < h_1(x_2).$$

We can treat  $h_2$  as an element of  $H_+[a, b]$  by the natural extension (assuming that  $h_2(x) = x$  outside of  $[h_1(x_1), h_1(x_2)]$ ).

The interval  $[h_1^{-1}(y_1), h_1^{-1}(y_2)]$  is a nonempty subinterval of  $[x_1, x_2]$ . Now observe that for every  $z \in [h_1^{-1}(y_1), h_1^{-1}(y_2)]$  we have  $z < h_1(x)$ . Therefore,  $h_2(z) = z$ . So, we get

$$h_1(h_2(z)) = h_1(z) \in [y_1, y_2],$$

while

$$h_2(h_1(z)) \in [h_2(y_1), h_2(y_2)].$$

Since  $y_2 < h_2(y_1)$ , we can conclude that  $h_2 \circ h_1 \neq h_1 \circ h_2$  and  $H_+[a, b]$  is nonabelian.  $\square$

**Definition 2.6.** (see, for example, [6, 28]) A ternary relation  $R \subseteq X^3$  on a set  $X$  is said to be a *cyclic ordering* if:

1.  $\begin{cases} a \neq b \neq c \neq a \\ (a, b, c) \notin R \end{cases} \Leftrightarrow (c, b, a) \in R.$
2.  $(a, b, c) \in R \Rightarrow (b, c, a) \in R.$
3.  $\begin{cases} (a, b, c) \in R \\ (a, c, d) \in R \end{cases} \Rightarrow (a, b, d) \in R.$

Let  $X$  be a topological space and  $R$  be a cyclic ordering on  $X$ . A homeomorphism  $f: X \rightarrow X$  is *orientation preserving* if  $f$  preserves  $R$ , meaning that  $(z, y, x) \in R$  implies  $(f(z), f(y), f(x)) \in R$ . The set of all such autohomeomorphisms is a subgroup of  $H(X)$  which we denote by  $H_+(X)$ .

### 3. Order-preserving homeomorphisms and $\alpha$ -minimality

Using some results of Nachbin we extend the ideas of Gartside and Glyn [22] to compact connected linearly ordered spaces (Theorem 3.4).

For the purposes of this section we fix the following notations. Let  $(X, \tau_\leq)$  be a compact LOTS with its unique compatible uniform structure  $\mu$  and denote  $s = \min X, t = \max X$ . For every  $f \in C(X)$  and  $\varepsilon > 0$  define

$$U_{f,\varepsilon} := \{(x, y) \in X \times X : |f(x) - f(y)| \leq \varepsilon\}.$$

Denote by  $C_+(X, [0, 1])$  the set of all continuous order-preserving maps  $f: X \rightarrow [0, 1]$ .

**Lemma 3.1.** (Nachbin [32]) *Let  $X$  be a compact LOTS.*

1.  $C_+(X, [0, 1])$  separates the points of  $X$ .
2. The family  $\{U_{f,\varepsilon} : f \in C_+(X, [0, 1]), \varepsilon > 0\}$  is a subbase of the uniformity  $\mu$  for every compact LOTS  $X$ .

*Proof.* (1) It is a fundamental result of Nachbin [32, p. 48 and 113].

(2) Use (1) and the following observation. For every compact space  $X$  and a point-separating family  $F$  of (uniformly) continuous functions  $X \rightarrow [0, 1]$ , the corresponding weak uniformity  $\mu_F$  on  $X$  is just the natural unique compatible uniformity  $\mu$  on  $X$ . The family of entourages  $\{U_{f,\varepsilon} : f \in F, \varepsilon > 0\}$  is a uniform subbase of  $\mu = \mu_F$ .  $\square$

**Definition 3.2.** Let  $\alpha \in \mu$  be an entourage. We say that a finite chain  $A := \{c_0, c_1, \dots, c_n\}$  in  $X$  is an  $\alpha$ -connected net if :

1.  $s = c_0 \leq c_1 \leq \dots \leq c_n = t$ ;
2.  $(x, y) \in \alpha$  for every  $x, y \in [c_i, c_{i+1}]$  and  $0 \leq i \leq n - 1$ .



Notation:  $A \in \Gamma(\alpha)$ .

Note that  $(x, y) \in \alpha^2$  for every  $x \in [c_k, c_{k+1}]$  and  $y \in [c_{k+1}, c_{k+2}]$ .

**Lemma 3.3.** *Let  $(X, \tau_\leq)$  be a compact LOTS with its unique compatible uniform structure  $\mu$ . The following are equivalent:*

1.  $X$  is connected;
2. for every  $\alpha \in \mu$  there exists an  $\alpha$ -connected net.

*Proof.* (1)  $\Rightarrow$  (2)

In the setting of Definition 3.2 every finite chain which contains an  $\alpha$ -connected net is also an  $\alpha$ -connected net. It follows that it is enough to verify the definition for entourages from any given uniform subbase of  $\mu$ . So, in our case, by Lemma 3.1, it is enough to check that there exists an  $\alpha$ -connected net for every  $\alpha = U_{f, \varepsilon}$ . We have to show that  $\Gamma(U_{f, \varepsilon})$  is nonempty for every  $f \in C_+(X, [0, 1])$  and every  $\varepsilon > 0$ .

Since  $X$  is connected and compact the continuous image  $f(X) \subseteq [0, 1]$  is a closed subinterval, say  $f(X) = [u, v]$ .

Fix  $n \in \mathbb{N}$  large enough such that  $\frac{v-u}{n} \leq \varepsilon$ . For every natural  $i$  with  $0 < i < n$  choose  $c_i \in X$  with  $f(c_i) = \frac{(v-u)i}{n} + u$  and  $c_0 = s$ ,  $c_n = t$ . Then

$$A := \{c_0, c_1, \dots, c_n\} \in \Gamma(U_{f, \varepsilon}).$$

Indeed, since  $f$  is order-preserving, for every  $x, y \in X$  with  $x, y \in [c_i, c_{i+1}]$  we have

$$f(x), f(y) \in [f(c_i), f(c_{i+1})].$$

So  $|f(x) - f(y)| \leq \frac{v-u}{n} \leq \varepsilon$ . Therefore,  $(x, y) \in \alpha = U_{f, \varepsilon}$ .

(2)  $\Rightarrow$  (1)

Assume to the contrary that  $X$  is not connected. Since  $X$  is a compact LOTS it follows that the order is not dense. That is, there exist  $a < b$  in  $X$  such that the interval  $(a, b)$  is empty. Then the function  $f: X \rightarrow [0, 1]$ , where  $f(x) = 0$  for  $x \leq a$  and  $f(x) = 1$  for  $b \leq x$  is continuous. Choose any  $0 < \varepsilon < 1$  and define  $\alpha := U_{f, \varepsilon} \in \mu$ . Then  $\Gamma(\alpha)$  is empty.  $\square$

Assertion (2) of the following theorem for  $X := [0, 1]$  generalizes a result of [22] mentioned above in Theorem 1.9. We modify the arguments of [22] and use Lemmas 1.5, 2.5 and 3.3.

**Theorem 3.4.** *Let  $(X, \tau_\leq)$  be a compact connected LOTS that satisfies the following condition:*

(A) *for every pair of elements  $a < b$  in  $X$  the group  $H_+[a, b]$  is nontrivial.*

*Then:*

- (1) *For the topological groups  $G = H_+(X)$  and  $G = H(X)$  the Zariski and Markov topologies coincide with the compact-open topology. That is,  $\mathfrak{Z}_G = \mathfrak{M}_G = \tau_{co}$ .*

(2) The topological groups  $H_+(X)$  and  $H(X)$  are  $a$ -minimal.

(3)  $X$  is  $aM_+$ -compact and  $aM$ -compact.

*Proof.* Assertion (2) follows from (1) by applying Lemma 1.5. By Definition 1.2 assertion (3) is a reformulation of (2). So it is enough to prove (1).

Below  $G$  denotes one of the groups  $H_+(X)$  or  $H(X)$ . Denote by  $\tau_{co}$  the (compact-open) topology on  $G$ . By Lemma 1.5 it is equivalent to show that  $\tau_{co} \subseteq \mathfrak{Z}_G$ .

For every interval  $(a, b) \subseteq X$  (with  $a < b$ ) the group  $H_+[a, b]$  is nontrivial (condition (A)) and thus, by Lemma 2.5, this group is nonabelian. Taking into account Lemmas 2.5 and 2.4 choose  $p, q \in H_+(X)$  such that  $pq \neq qp$  and  $p(x) = q(x) = x$  for every  $x \notin (a, b)$ . Define

$$T(a, b) := \{g \in G : gpg^{-1} \text{ does not commute with } q\}. \quad (3.1)$$

**Claim 3.5.**  $e \in T(a, b) \in \mathfrak{Z}_G$ .

*Proof.* Indeed, rewrite the definition of  $T(a, b)$  to obtain

$$T(a, b) = \{g \in G : (gpg^{-1})q(gpg^{-1})^{-1}q^{-1} \neq e\}$$

and use Definition 1.4 to conclude that  $T(a, b) \in \mathfrak{Z}_G$ . The fact that  $e \in T(a, b)$  is trivial by the choice of  $p, q$ .  $\square$

**Claim 3.6.** For every  $g \in T(a, b)$  there exists  $x \in (a, b)$  such that  $g(x) \in (a, b)$ . That is,  $g(a, b) \cap (a, b) \neq \emptyset \quad \forall g \in T(a, b)$ .

*Proof.* Assuming the contrary, there exists  $g \in T(a, b)$  such that  $g(a, b) \cap (a, b) = \emptyset$ . Equivalently,  $(a, b) \cap g^{-1}(a, b) = \emptyset$ . Hence,  $g^{-1}(x) \notin (a, b)$  for every  $x \in (a, b)$ . By the choice of  $p$  we have  $pg^{-1}(x) = g^{-1}(x)$  and so  $gpg^{-1}(x) = x$  for every  $x \in (a, b)$ . On the other hand,  $q(x) = x$  for every  $x \in X \setminus (a, b)$  (by the choice of  $q$ ). It follows that  $gpg^{-1}$  and  $q$  commute, which contradicts the definition of  $T(a, b)$  in (3.1).  $\square$

Let  $\alpha$  be the collection of all finite intersections of  $T(a, b)$ 's. By Claim 3.5 (using that  $\mathfrak{Z}_G$  is a topology) we obtain  $\alpha \subseteq \mathfrak{Z}_G$ . Both  $\tau_{co}$  and  $\mathfrak{Z}_G$  are completely determined by the neighbourhood base at  $e \in G$ . So, in order to see that  $\tau_{co} \subseteq \mathfrak{Z}_G$  it suffices to show the following.

**Claim 3.7.** Every open neighbourhood  $U$  of  $e$  in  $G$ , with the compact-open topology  $\tau_{co}$ , contains an element  $T$  from  $\alpha$ .

*Proof.* Let  $\mu$  be the unique compatible uniformity on  $X$ . A basic neighbourhood of  $e$  has the form:

$$O_\varepsilon := \{g \in G : (g(x), x) \in \varepsilon \quad \forall x \in X\},$$

where  $\varepsilon \in \mu$ . Choose a symmetric entourage  $\varepsilon_1 \in \mu$  such that  $\varepsilon_1^2 \subseteq \varepsilon$ . For  $\varepsilon_1$  by Lemma 3.3 choose an  $\varepsilon_1$ -connected net

$$c_0 < c_1 < \cdots < c_n$$

of  $X$ . We can suppose that  $X$  is nontrivial and  $n > 0$ .

By Equation 3.1, we have the corresponding  $T(c_i, c_{i+1}) \subseteq G$  for every index  $0 \leq i \leq n-1$ . Define

$$T := \bigcap_{i=0}^{n-1} T(c_i, c_{i+1}).$$

Now it is enough to show:

$$T \subseteq O_\varepsilon. \quad (3.2)$$

Assuming the contrary let  $h \in T$  but  $h \notin O_\varepsilon$ . Then there exists  $x \in X$  such that  $(h(x), x) \notin \varepsilon$ . Pick minimal index  $k$  between 0 and  $n-1$  such that  $x \in [c_k, c_{k+1}]$ . Then by a remark after Definition 3.2 we have  $(x, y) \in \varepsilon_1^2 \subseteq \varepsilon$  for every  $y \in [c_{k-1}, c_{k+2}]$ . If  $k = 0$ , we replace  $c_{k-1}$  by  $c_0$ . Similarly, we replace  $c_{k+2}$  by  $c_n$  if  $k = n-1$ .

Hence,

$$h(x) \in X \setminus [c_{k-1}, c_{k+2}] = [c_0, c_{k-1}) \cup (c_{k+2}, c_n]. \quad (3.3)$$

Note that one of the intervals in the union can be empty.

From Claim 3.6 for every index  $0 \leq i \leq n-1$  choose  $x_i$  such that

$$x_i, h(x_i) \in (c_i, c_{i+1}). \quad (3.4)$$

We show that there is no such  $h \in G$ . By Lemma 2.3 any autohomeomorphism  $h \in H(X)$  is either order-preserving or order-reversing. By Equation 3.4 we have  $h(x_i) < h(x_{i+1})$ , where  $x_i < x_{i+1}$ . So,  $h$  can be only order-preserving.

Now, we show that  $h$  is not order-preserving. Indeed, we have the following two cases:

$$(1) \quad h(x) \in (c_{k+2}, c_n].$$

Then,  $h(x_{k+1}) < h(x)$ , while  $x < x_{k+1}$ .

$$(2) \quad h(x) \in [c_0, c_{k-1}).$$

Then,  $h(x) < h(x_{k-1})$ , while  $x_{k-1} < x$ .

In both cases we get a contradiction. This completes the proof of Equation 3.2 and hence of our theorem.  $\square$

$\square$

**Corollary 3.8.** *Let  $(X, \tau_\leq)$  be a compact connected LOTS that satisfies the following condition:*

(C) for every pair of elements  $a < b$  in  $X$  there exist  $c, d \in X$  with  $a \leq c < d \leq b$  such that  $[c, d]$  is separable (equivalently, the subspace  $[c, d] \subseteq X$  is homeomorphic to the real unit interval  $[0, 1]$ ).

Then  $\mathfrak{Z}_G = \mathfrak{M}_G = \tau_{co}$  and the groups  $G = H_+(X)$ ,  $G = H(X)$  are  $a$ -minimal (that is,  $X$  is  $aM_+$ -compact and  $aM$ -compact).

*Proof.* Recall (see, for example, [19, Exercise 6.3.2]) that a separable linearly ordered continuum is homeomorphic to  $[0, 1]$ . It is well known and easy to see that  $H_+[0, 1]$  is nonabelian (Section 4.4). Also, up to the inversion, there exists only one linear order on  $[0, 1]$  inducing the natural topology [28, Cor. 4.1]. We see that (C) implies that  $H_+[c, d]$  (being a copy of  $H_+[0, 1]$ ) is nonabelian. So, we can apply Theorem 3.4.  $\square$

## 4. Some examples

### 4.1. Not every compact LOTS is $M_+$ -compact

The following example shows that  $H_+(X)$  is not necessarily minimal.

*Example 4.1.* Denote by  $\mathbb{Z}^*$  the two-point compactification of  $\mathbb{Z}$ . One can easily verify that  $H_+(\mathbb{Z}^*)$  is a discrete copy of  $\mathbb{Z}$  and thus not minimal. That is, the compact LOTS  $\mathbb{Z}^*$  is not  $M_+$ -compact. Note that  $\mathbb{Z}^*$  is also not  $M$ -compact as it directly follows from [11, Theorem 4.25].

### 4.2. Rigid ordered compact spaces

Let us say that a topological space  $X$  is  $H$ -rigid if the group  $H(X)$  is trivial. Similarly, let us say that a linearly ordered space  $X$  is  $H_+$ -rigid if the group  $H_+(X)$  is trivial. Certainly, if  $X$  is  $H$ -rigid then it is also  $H_+$ -rigid. There are many known examples of  $H$ -rigid compact spaces, and in particular of compact ordered  $H$ -rigid spaces. Most of the examples of the latter kind (Jonsson, Rieger, de Groot-Maurice) are zero-dimensional. It seems that the first ("naive") example of a nontrivial *connected* compact ordered  $H$ -rigid space was constructed by Hart and van Mill [26]. Note also that, under the *diamond principle*, there exists an  $H$ -rigid Suslin continuum (Jensen, see in [39, p. 268]).

Using results of [26], one may show the following.

**Proposition 4.2.** *There exists an ordered continuum  $X$  with  $H_+(X) = H(X) = \mathbb{Z}$ , a discrete copy of the integers.*

So, we get a *connected* compact LOTS  $X$  such that  $H_+(X)$  is not minimal (or,  $X$  is not  $M_+$ -compact). Hence, Theorem 3.4 does not remain true for general ordered continua.

We sketch the proof of Proposition 4.2. Let  $L := [a, b]$  be the ordered continuum constructed in [26, Section 5]. This space has very few continuous selfmaps. Any continuous map  $f: L \rightarrow L$  is a *canonical retraction*. That is, there exists a pair  $u \leq v \in L$  such that

$$f(x) = u \quad \forall x \leq u, \quad f(x) = x \quad \forall u \leq x \leq v, \quad f(x) = v \quad \forall x \geq v.$$

In particular,  $L$  is  $H$ -rigid. Moreover, for every topological embedding  $f: L \rightarrow L$  we have  $f = id$ . Note also the following *special property* which we use below: if  $f(a) = a$  then either  $f(x) = x$  for every  $x \in U$  on some neighbourhood  $U$  of  $a$ , or  $f$  is the constant map  $f(x) = a$  for every  $x \in L$ .

Now the desired continuum  $X$  will be the two point compactification of some locally compact connected LOTS  $Y$ , the "long  $L$ ". More precisely, the corresponding linearly ordered set  $Y$  is the lexicographically ordered set  $\mathbb{Z} \times [a, b)$ . Endow  $Y$  with its usual interval topology. Every subinterval in  $Y$  of the form

$$L_n := [(n, a), (n+1, a)] = \{(n, x) : x \in [a, b)\} \cup \{(n+1, a)\}$$

is naturally order isomorphic with  $L$  for every  $n \in \mathbb{Z}$ . Our aim is to show that  $H_+(X) = H(X) = \mathbb{Z}$ . First of all we have a naturally defined (shift) homeomorphism  $\sigma: X \rightarrow X$  where  $\sigma(n, x) = (n+1, x)$  for every  $n \in \mathbb{Z}, x \in [a, b)$ . We claim that any other homeomorphism  $f: X \rightarrow X$  is  $\sigma^k$  (the  $k$ -th iteration) for some  $k \in \mathbb{Z}$ . Indeed, if  $f(L_0) \subseteq L_k$  for some  $k \in \mathbb{Z}$  then  $f(L_0) = L_k$ . Moreover it is easy to see that  $f = \sigma^k$ . Now assume that  $f(L_0) \subseteq L_k$  is not true for every  $k \in \mathbb{Z}$ . Then there exists  $k \in \mathbb{N}$  such that  $f(0, a) < (k, a) < f(0, b)$ . Consider the retraction

$$h: X \rightarrow X, h(z) = (k, a) \quad \forall z \leq (k, a), \text{ and } h(z) = z \quad \forall z > (k, a).$$

Then the composition  $h \circ f$  restricted on  $L_0$  defines a nonconstant continuous map  $L_0 \rightarrow L_k$  which moves  $(0, a)$  to  $(k, a)$ . This induces a continuous nonconstant selfmap  $q: L \rightarrow L$  such that  $q(x) = a \quad \forall x \in U$  for some neighborhood  $U$  of  $a$ . By the *special property* of  $L$  mentioned above, we get a contradiction. These arguments show that algebraically  $H_+(X) = H(X) = \mathbb{Z}$ . Finally observe that  $H_+(X)$  is discrete in the compact-open topology.

#### 4.3. The Ordinal Space

For every ordinal number  $\kappa$  the space  $[0, \kappa]$  is a compact LOTS. This space is scattered and hence not connected for every  $\kappa > 0$ . Nonetheless, one can show that  $H_+[0, \kappa]$  is trivial (hence  $a$ -minimal). We start by noting that  $[0, \kappa]$  is certainly a well-ordered set.

**Lemma 4.3.** [7, Corollary 4.1.9] *If two well-ordered sets  $A$  and  $B$  are order-isomorphic, then the isomorphism is unique.*

It follows from Lemma 4.3 that the identity is the only order-preserving automorphism of a well-ordered set.

**Corollary 4.4.** *Every well-ordered compact LOTS  $X$  (e.g., the ordinal space  $X = [0, \kappa]$ ) is  $H_+$ -rigid. That is,  $H_+(X) = \{e\}$  (thus  $X$  is  $aM_+$ -compact).*

This example shows that the condition of Theorem 3.4 is not necessary.

#### 4.4. The Unit Interval

The group  $H_+[0, 1]$  (and, hence, also any  $H_+[a, b]$  for every two reals  $a < b$ ) is not abelian. Take, for example, the following pair  $f, h$  of noncommuting elements. Define  $f(x) = x^2$ ,  $h(x) = 0.5x$  for  $0 \leq x \leq 0.5$  and  $h(x) = 1.5x - 0.5$  for  $0.5 \leq x \leq 1$ . So, the continuum  $[0, 1]$  clearly satisfies the conditions of Theorem 3.4. Therefore, the groups  $H_+[0, 1]$  and  $H[0, 1]$  are  $a$ -minimal.

#### 4.5. The Ordered Square

Let  $I = [0, 1]$  and define the lexicographic order on  $I \times I$ . Then  $\mathcal{I}^2 = (I \times I, \tau_{\leq})$ , the unit square with the order topology, is a compact and not metrizable space. We show that it satisfies the conditions of Corollary 3.8. It is connected (see [37, Section 48]). As to the second condition, let  $K = [(a_1, b_1), (a_2, b_2)] \subseteq \mathcal{I}^2$  be a closed interval. If  $a_1 = a_2$  then  $K$  is homeomorphic to  $[0, 1] \subseteq \mathbb{R}$ . Otherwise, if  $a_1 < a_2$ ,  $K$  contains an interval homeomorphic to  $[0, 1] \subseteq \mathbb{R}$  (for example  $[(\frac{a_1+a_2}{2}, 0), (\frac{a_1+a_2}{2}, 1)]$ ). Thus condition (C) of Corollary 3.8 is satisfied. Hence,  $H_+(\mathcal{I}^2)$  and  $H(\mathcal{I}^2)$  are  $a$ -minimal (and  $\mathcal{I}^2$  is both  $aM_+$ -compact and  $aM$ -compact).

#### 4.6. The Extended Long Line

Let  $\mathcal{L}$  be the set  $[0, \omega_1) \times [0, 1)$  where  $\omega_1$  is the least uncountable ordinal. Considering  $\mathcal{L}$  with the lexicographic order, the set  $\mathcal{L}$  with the topology induced by this order is called *the long line*. Let  $\mathcal{L}^* = \mathcal{L} \cup \{\omega_1\}$  and extend the ordering on  $\mathcal{L}$  to  $\mathcal{L}^*$  by letting  $a < \omega_1$  for all  $a \in \mathcal{L}$ . The space  $\mathcal{L}^*$  with the order topology is a compact space called *the extended long line*. In fact,  $\mathcal{L}^*$  is the one point compactification of  $\mathcal{L}$ .

Several properties of this space can be found in [27], [31] and [35]. The extended long line satisfies the conditions of Corollary 3.8. Indeed, it is well known that  $\mathcal{L}^*$  is a compact connected LOTS. Also,  $\mathcal{L}$  (the long line) is locally homeomorphic (by an order-preserving homeomorphism) to the interval  $(0, 1)$ . In case the interval in question is of the form  $[a, \omega_1]$ , we can verify condition (C) for a subinterval  $[a, b]$  of  $[a, \omega_1]$ , where  $b \neq \omega_1$ . So,  $H_+(\mathcal{L}^*)$  and  $H(\mathcal{L}^*)$  are  $a$ -minimal. Hence,  $\mathcal{L}^*$  is both  $aM_+$ -compact and  $aM$ -compact.

#### 4.7. The Circle

Recall the definition of the natural cyclic ordering (Definition 2.6) on the unit circle  $\mathbb{S}^1$ . Identify  $\mathbb{S}^1$ , as a set, with  $[0, 1)$  and define a ternary relation  $R \subseteq [0, 1)^3$  as follows:  $(z, y, x) \in R$  if and only if  $(x - y)(y - z)(x - z) > 0$ . Denote by  $H_+(\mathbb{S}^1)$  the Polish group of all orientation preserving homeomorphisms of the circle  $\mathbb{S}^1$ .

The arguments of Theorem 3.4 (or, of [22, Theorem 1]) can be easily modified for the circle  $\mathbb{S}^1$ , hence:

**Theorem 4.5.** *The group  $H_+(\mathbb{S}^1)$  is  $a$ -minimal.*

Note that the coset space  $H_+(\mathbb{S}^1)/\text{St}(z)$  is naturally homeomorphic to the circle, where  $\text{St}(z)$  is the stabilizer group of any given  $z \in \mathbb{S}^1$ . So the minimality of  $H_+(\mathbb{S}^1)$  can be derived from the minimality of  $H_+[0, 1]$  using Lemma 2.2 and the fact that  $\text{St}(z)$  is topologically isomorphic to  $H_+[0, 1]$ .

Since  $H_+(\mathbb{S}^1)$  is a closed normal subgroup of  $H(\mathbb{S}^1)$ , and  $H(\mathbb{S}^1)/H_+(\mathbb{S}^1) \cong \mathbb{Z}_2$ , we can use Lemma 2.2 one more time to deduce the minimality of  $H(\mathbb{S}^1)$ .

A Hausdorff topological group is *totally minimal* if every Hausdorff quotient is minimal [13]. Every minimal algebraically (or, at least, topologically) simple minimal group is totally minimal.  $H_+(\mathbb{S}^1)$  is algebraically simple as can be seen (for example) in [36, 24]. Although the group  $H_+[0, 1]$  is not algebraically simple, it is topologically simple. Indeed, by [20, Theorem 14],  $H_+[0, 1]$  has exactly five normal subgroups:  $\{e\}$ ,  $H_+[0, 1]$ ,  $Q_1$ ,  $Q_0$ ,  $Q := Q_0 \cap Q_1$ . It is easy to see that  $Q$  is dense in  $H_+[0, 1]$ . This yields that  $H_+[0, 1]$  is topologically simple.

**Corollary 4.6.**  $H_+(\mathbb{S}^1)$  and  $H_+[0, 1]$  are totally minimal groups.

## 5. Some questions

A more general version of Question 1.3 is the following.

*Question 5.1.* When appropriate subgroups  $G$  of  $H(X)$  (say, the automorphism groups of some structures on  $X$ ) are minimal (a-minimal) ?

We already know that the Cantor cube  $2^\omega$  is  $M$ -compact ([21]).

*Question 5.2.*

1. Is the Cantor cube  $2^\omega$   $aM$ -compact ?
2. Is the Cantor set  $X \subseteq [0, 1]$ , as a linearly ordered compact LOTS,  $M_+$ -compact ?  $aM_+$ -compact ?
3. Is the space  $2^\lambda$   $M$ -compact (or,  $aM$ -compact) for every cardinal  $\lambda$  ?

*Question 5.3.*

1. Is it true that every  $M$ -compact space is also  $aM$ -compact ?
2. Is it true that every linearly ordered connected  $M_+$ -compact space is  $aM_+$ -compact ?
3. Is it true that for ordered continua condition (A) of Theorem 3.4 is really weaker than condition (C) of Corollary 3.8 ?

In view of Markov's Question 1.6 and Theorem 3.4 we have several good reasons to pose the following question.

*Question 5.4.* For what compact (linearly ordered) spaces  $X$  the Markov and Zariski topologies coincide on the group  $G = H(X)$  (resp.,  $G = H_+(X)$ ) ?

Various properties of the homeomorphism group  $H(X)$  of several important 1-dimensional continua  $X$  were intensively studied from several points of view. Among others is the case where  $X$  is the *pseudo-arc* or the *Lelek Fan*. About the latter case, see, for example, the very recent works of Bartošova-Kwiatkowska [3, 4] and Ben Yaacov-Tsankov [5].

*Question 5.5.* Let  $X$  be the pseudo-arc or the Lelek fan. Is it true that  $H(X)$  is minimal ?  $a$ -minimal ?

It is well known that the pseudo-arc is a homogeneous compactum. So the previous question is related to Stoyanov's Question 1.1. Another property of the pseudo-arc is that it is a chainable continuum. Recall that a compact space  $X$  is *chainable* if every (finite) open cover  $\varepsilon$  has a finite open refinement  $\alpha$  that is an  $\varepsilon$ -small chain, that is,  $\alpha = \{O_1, \dots, O_n\}$ , where  $O_i \cap O_j \neq \emptyset \Leftrightarrow |i - j| \leq 1$  and every  $O_i$  is  $\varepsilon$ -small. Every linearly ordered continuum is chainable. This follows, for example, by Lemma 3.3. Therefore, it would be interesting to extend Theorem 3.4 to some broader class of chainable continua.

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